SELF-SIMILAR PROBLEMS OF MIXING OF A VISCOUS FLUID

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The present paper is concerned with one of the simplest problems of mixing, namely, the mixing of a viscous incompressible fluid. The complete boundary value problem is analyzed using the group-theoretic approach and it is shown, that various problems of viscous mixing (discharge of fluid, flow within a wake etc.) are obtained as particular cases corresponding to some definite values of the constant m appearing in the solution of the boundary value problem, where the solution is invariant under the admissible group of transformations

Problems of mixing of viscous fluids are widely studied (see e.g. [1 – 4]). The following characteristic feature of the physical statement of the problem emerges from these studies: it is the assumption of existence of a "narrow" zone of mixing extending along the stream, within which some of the flow parameters (longitudinal velocity, temperature, concentration, etc.) vary sharply in the transverse direction, while other (e.g. pressure) change significantly only in the direction of flow. Such zones of mixing appear in the presence of a sharp change in the values of one or several flow parameters and represent the region of diffusion of this change. This region increases according to some law in the direction of the longitudinal velocity and the flow in such narrow zones can be described by the boundary layer equations.

1. Statement of the problem. We shall consider the upper half of the plane flow with mixing of a viscous incompressible fluid, symmetric with respect to the horizontal axis of the flow and described by the following boundary layer equations

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1.1}$$

where $u(x,y)$ and $v(x, y)$ are, respectively, the horizontal and vertical velocity, ν is the kinematic viscosity coefficient, while x and y are the Cartesian coordinates. The velocities u and v should satisfy the following boundary conditions

$$\lim\,[u(x, y)/u_0(y)] = 1 \quad \text{when } x = \text{const},\ y \to \infty \tag{1.2}$$

$$\lim\,[u(x, y)/u_0(y)] = 1 \quad \text{when } x \to 0,\ y = \text{const} \tag{1.3}$$

$$(\partial u/\partial y)_{y=0} = 0, \qquad (v)_{y=0} = 0 \tag{1.4}$$

where $u_0(y)$ denotes the horizontal velocity profile in the cross section $x = 0$. Without making any definite assumptions about $u_0(y)$, we shall require the solution of the problem (1.1) – (1.4) to be self-similar.

2. Group-theoretic analysis of the equations and reduction of the boundary value problem to the Cauchy's problem for the ordinary equation. First we shall find the self-similar solution of (1.1) satisfying (1.4). After that we shall show, which functions $u_0(y)$ correspond to one or the other self-similar solution, exhausting all the possible functions $u_0(y)$ which can appear in the statement of the problem of viscous mixing in the case when the axis $y=0$ is the symmetry axis and the fluid is incompressible.

We know [5] that the self-similar solution is invariant with respect to any similarity group transformations (in this case it will be a one-parameter group) admitted by the system (1.1). Paper [6] shows that the system (1.1) admits two one-parameter similarity groups whose arbitrary superposition has the form

$$u_1 = C^{\alpha-2\beta} u, \quad v_1 = C^{-\beta} v, \quad x_1 = C^{\alpha} x, \quad y_1 = C^{\beta} y \tag{2.1}$$

where α and β are arbitrary constants $(-\infty < \alpha, \beta < \infty)$. Any self-similar solution of (1.1) is invariant with respect to a group of the form (2.1) with some fixed values of α and β, and has the form

$$u = \varphi'(\lambda)\, x^{\frac{m}{m+2}}, \quad v = \psi(\lambda)\, x^{-\frac{1}{m+2}}, \quad \lambda = y x^{-\frac{1}{m+2}} \quad \left(m = \frac{\alpha}{\beta} - 2\right) \tag{2.2}$$

where the prime denotes a derivative with respect to the self-similar variable λ. Integrating the equations of continuity we obtain the following relation between $\psi(\lambda)$ and $\varphi(\lambda)$

$$\psi(\lambda) = \frac{1}{m+2}\lambda\varphi' - \frac{m+1}{m+2}\varphi \tag{2.3}$$

Solution corresponding to the case when $\alpha = 0$ in (2.1) is not included in the relation (2.2). It was obtained in [6] and we shall not consider it here since it is uninteresting.

Inserting (2.2) into (1.1) we obtain the following ordinary differential equation in $\varphi(\lambda)$

$$\nu\varphi''' + \frac{m+1}{m+2}\varphi\varphi'' - \frac{m}{m+2}\varphi'^2 = 0 \qquad (\nu > 0,\ m > -2) \tag{2.4}$$

with the boundary conditions

$$\varphi(0) = 0, \quad \varphi''(0) = 0 \tag{2.5}$$

obtained from (1.4) and (2.2), and with

$$\varphi'(0) = \gamma \qquad (\gamma > 0) \tag{2.6}$$

taken as the third condition.

Each solution of the problem (2.4) – (2.6) with some fixed values of ν, γ and m generates, by (2.2), a solution $u(x, y)$, $v(x, y)$ of the system (1.1) satisfying the conditions (1.4). We can easily see from (1.2), (1.3) and (2.2) that the character of the flow in the physical plane is governed by the asymptotic behavior of the solution $\varphi(\lambda)$ of the problem (2.4) – (2.6) as $\lambda \to \infty$. Indeed, when we know the asymptotic behavior of the solution $\varphi(\lambda)$ as $\lambda \to \infty$, we can always find such a function $u_0(y)$ for which the relations (1.2) and (1.3) hold, i.e. we can define the problem of mixing completely.

It can easily be shown that the parameters ν and γ appearing in the formulation of the problem (2.4) – (2.6) will not be essential.

Indeed, introducing the function $\varphi_1(\lambda) = \nu^{-1}\varphi(\lambda)$ we can eliminate ν from (2.4), and subsequent substitution

$$\varphi_2(\lambda_1) = \left(\frac{\gamma}{\nu}\right)^{-1/2}\varphi_1(\lambda), \qquad \lambda_1 = \left(\frac{\gamma}{\nu}\right)^{1/2}\lambda \tag{2.7}$$

will reduce the problem for $\varphi_2(\lambda_1)$ to

$$\varphi_2''' + \frac{m+1}{m+2}\varphi_2\varphi_2'' - \frac{m}{m+2}\varphi_2'^2 = 0, \quad \varphi_2(0) = \varphi_2''(0) = 0, \quad \varphi_2'(0) = 1 \quad (2.8)$$

Thus the solution of (2.4) – (2.6) and its asymptotic behavior can easily be obtained from the solution of the asymptotic behavior of the solution of (2.8), using the relations

$$\varphi(\lambda) = (\nu\gamma)^{1/2}\varphi_2(\lambda_1), \qquad \lambda = \left(\frac{\nu}{\gamma}\right)^{1/2}\lambda_1 \quad (2.9)$$

When considering the problems of mixing, we usually introduce two integral characteristics: the impulse I_0 and the flux I_1 which can be written in the plane case and with the symmetry taken into account, as

$$I_0 = 2\int_0^y u^2 dy = 2x^{\frac{2m+1}{m+2}}\nu^{1/2}\gamma^{3/2}\int_0^{\lambda_1}\varphi_2'^2(\lambda_1)\,d\lambda_1 \quad (2.10)$$

$$I_1 = 2\int_0^y u dy = 2x^{\frac{m+1}{m+2}}(\nu\gamma)^{\frac{1}{2}}\varphi_2(\lambda_1) \quad (2.11)$$

Relations (2.10) and (2.11) show that both, I_0 and I_1, remain, for some value of m, constant along the x-axis.

3. Analysis of some real flows. From (2.2) we see that the case $m > 0$ corresponds to the problems dealing with mixing of two streams (or the flow in the wake behind a body) and that the velocity $u(x, y)$ along the axis of symmetry of flow increases, while the case $0 > m > -2$ corresponds to the problems of mixing during the efflux of fluid when the velocity along the axis of symmetry decreases.

The solution of (2.8) lends itself to the analytic treatment at four values of m, namely $m = 0$, $m = -0.5$, $m = -1$, $m = 1$. Let us consider the case $m = -0.5$.

a) The case $m = -0.5$ has been studied exhaustively (see e.g. [7]). It corresponds to the case of a submerged stream. The problem (2.8) becomes, in this case.

$$\varphi_2''' + {}^1/_3\,\varphi_2\varphi_2'' + {}^1/_3\varphi_2'^2 = 0, \quad \varphi_2(0) = \varphi_2''(0) = 0, \quad \varphi_2'(0) = 1 \quad (3.1)$$

which can be easily integrated. Arbitrary constants are determined from the initial conditions, and the resulting solution has the form

$$\varphi_2(\lambda_1) = \sqrt{6}\,\mathrm{th}\frac{\lambda_1}{\sqrt{6}} \quad (3.2)$$

Solution $\varphi_2(\lambda_1)$ and its derivative $\varphi_2'(\lambda_1)$ behave as follows:

$$\varphi_2(\lambda_1) \to \sqrt{6}, \quad \varphi_2'(\lambda_1) = 2\left[1 + \mathrm{ch}\left(\frac{2}{\sqrt{6}}\lambda_1\right)\right]^{-1} \to 0 \ \text{ when } \lambda_1 \to \infty \quad (3.3)$$

From (2.2), (2.9) and (3.3) we find, that

$$u(x, y) = x^{-1/3}\gamma\varphi_2'(\lambda_1) \to \begin{cases} 0 \ \text{ when } x = \text{const}, & y \to \infty \\ 0 \ \text{ when } x \to 0, & y = \text{const} \\ \infty \ \text{ when } x \to 0, & y = 0 \end{cases} \quad (3.4)$$

and in accordance with (1.2) and (1.3) we obtain

$$u_0(y) = 0 \ \text{when } y \neq 0, \qquad u_0(y) = \infty \ \text{when } y = 0 \quad (3.5)$$

It can easily be confirmed that the condition of the conservation of impulse along the x-axis characteristic for the submerged stream holds also in this case

$$I_0 = 2\nu^{1/2}\gamma^{3/2}\int_0^\infty \varphi_2'^2(\lambda_1)\,d\lambda_1 = 2\nu^{1/2}\gamma^{3/2} i_0 = \text{const} \qquad (3.6)$$

From (3.6) we see that if I_0 is given and the integral i_0 is bounded, then the constant γ can easily be obtained since i_0 can be found from the known solution $\varphi_2(\lambda_1)$, while ν is a known material parameter of the fluid. Thus in the present case γ is completely defined by the impulse imparted to the fluid particles at the point $x = 0$, $y = 0$ during the unit time.

b) Let us consider the case $m = 1$. The problem (2.8) is now equivalent to

$$\varphi_2^{IV} + {}^2/_3\,\varphi_2\varphi_2''' = 0, \quad \varphi_2(0) = \varphi_2''(0) = 0, \quad \varphi_2'(0) = 1, \quad \varphi_2'''(0) = {}^1/_3 \qquad (3.7)$$

Although (3.7) cannot be solved by analytical methods, numerical integration can always be employed.

We can, however, use the method given by Weyll in [8] to obtain the asymptotic behavior of the solution $\varphi_2(\lambda_1)$, and consequently, to find the form of $u_0(y)$.

From (3.7) we have

$$g(\lambda_1) = \Phi(g) = \exp\left(-\frac{\lambda_1^2}{3}\right)\exp\left[-\frac{1}{27}\int_0^{\lambda_1}(\lambda_1-\xi)^3 g(\xi)\,d\xi\right], \quad g(\lambda_1) = 3\varphi_2'''(\lambda_1) \qquad (3.8)$$

This integral equation can be solved by the method of successive approximations according to the scheme

$$g_0(\lambda_1) \equiv 1, \quad g_1(\lambda_1) = \Phi(g_0), \ldots, \ g_{i+1}(\lambda_1) = \Phi(g_i), \ldots$$

We shall prove that the sequence $\{g_i\}$ converges and find $\lim g_i(\lambda_1)$ as $i \to \infty$.

Obviously, if $q(\xi) \geqslant h(\xi)$ $(0 \leqslant \xi < +\infty)$, then $\Phi(q) \leqslant \Phi(h)$ $(0 \leqslant \xi < +\infty)$ and

$$g_0(\lambda_1) \geqslant g_1(\lambda_1), \qquad g_0(\lambda_1) \geqslant g_2(\lambda_1)$$

From this it follows that

$$g_0 \geqslant g_2 \geqslant g_4 \geqslant \ldots, \quad g_1 \leqslant g_3 \leqslant g_5 \leqslant \ldots, \quad g_{2i} \geqslant g_{2k+1} \quad (i, k = 0, 1, 2, \ldots) \qquad (3.9)$$

Let $0 < h(\xi) < q(\xi) \leqslant 1$ $(0 \leqslant \xi < +\infty)$ and let us put

$$\sup\,[q(\xi) - h(\xi)] = A \qquad (\xi \geqslant 0)$$

Then

$$\Phi(h) - \Phi(q) = \exp\frac{-\lambda_1^2}{3}\exp\left(-\frac{1}{27}\int_0^{\lambda_1}(\lambda_1-\xi)^3 h(\xi)\,d\xi\right) \times$$

$$\times\left[1 - \exp\left(-\frac{1}{27}\int_0^{\lambda_1}(\lambda_1-\xi)^3[q(\xi)-h(\xi)]\,d\xi\right)\right] \leqslant$$

$$\leqslant \exp\frac{-\lambda_1^2}{3}\left[1-\exp\left(-\frac{A\lambda_1^4}{108}\right)\right] \leqslant \exp\frac{-\lambda_1^2}{3}\,\frac{A\lambda_1^4}{108} \leqslant \frac{A}{3e^2} \qquad (3.10)$$

Relations (3.9) and (3.10) infer that the sequence $\{g_i\}$ $(i = 0, 1, 2, \ldots)$ converges uniformly on $[0, +\infty)$. Obviously

$$\varphi_2''(\lambda)_1 = {}^1/_3 \lim g_i(\lambda_1) \quad (i \to \infty), \qquad {}^1/_3 g_1(\lambda_1) \leqslant \varphi_2'''(\lambda_1) \leqslant {}^1/_3 g_2(\lambda_1) \quad (\lambda_1 \geqslant 0) \qquad (3.11)$$

$$\frac{1}{3}\exp\left(-\frac{\lambda_1^2}{3} - \frac{\lambda_1^4}{108}\right) \leqslant \varphi_2'''(\lambda_1) \leqslant$$

$$\leqslant \frac{1}{3}\exp\left(-\frac{\lambda_1^2}{3}\right)\exp\left[-\frac{1}{27}\int_0^{\lambda_1}(\lambda_1-\xi)^3\exp\left(-\frac{\xi^2}{3}-\frac{\xi^4}{108}\right)d\xi\right] \qquad (3.12)$$

From (3.12) it follows that at large λ_1:

$$\varphi_2'''(\lambda_1) = o(e^{-\lambda_1^2}) \tag{3.13}$$

Thus when $\lambda_1 \to \infty$, we have

$$\varphi_2''(\lambda_1) = a + o(e^{-\lambda_1^2}) \tag{3.14}$$

$$\varphi_2'(\lambda_1) \doteq a\lambda_1 + b + o(e^{-\lambda_1^2}) \tag{3.15}$$

$$\varphi_2(\lambda_1) = {}^1/_2 a\lambda_1^2 + b\lambda_1 + d + o(e^{-\lambda_1^2}) \tag{3.16}$$

where a, b and d are some positive constants whose values can be obtained by numerical methods.

From (2.2), (2.9) and (3.15) it follows, that for large y and $x > 0$

$$u(x, y) = x^{1/3}\gamma\varphi_2' \to \nu^{1/2}\gamma^{3/2}ay + \gamma b x^{1/3} + o\left(\exp\left[-\frac{\gamma}{\nu}\frac{y^2}{x^{2/3}}\right]\right) \tag{3.17}$$

which, together with the symmetry of the flow implies, that $u_0(y)$ will have the form

$$u_0(y) = \nu^{1/2}\gamma^{3/2} a\,|y| \tag{3.18}$$

Hence the value of γ can easily be found, provided that the derivative $\partial u_0(y)/\partial y$, i.e. the slope of the velocity profile at the cross section $x = 0$, is given.

c) The case $m = 0$ is trivial. Indeed in this case (2.8) becomes

$$\varphi_2''' + {}^1/_2\varphi_2\varphi_2'' = 0 \tag{3.19}$$

and $\varphi_2 = \lambda_1$ is the solution satisfying the initial conditions. The velocities will be $u(x, y) = \gamma\varphi_2'(\lambda_1) = \gamma = \text{const}$, and $v(x, y) = 0$, i.e. we shall have a plane parallel flow.

d) When $m = -1$, we can use the function $w = \varphi_2'(\lambda_1)$ to write the problem (2.11) in the form

$$w''' + w^2 = 0, \quad w(0) = 1, \quad w'(0) = 0 \tag{3.20}$$

which on integration yields

$$\lambda_1 = -\int_1^w \frac{d\xi}{\sqrt{{}^2/_3(1-\xi^3)}} \tag{3.21}$$

The latter formula shows that there exists λ_1^0, for which $w = 0$. From (2.3) we find that $\psi(\lambda_1^0) = 0$. Thus u and v become zero when $\lambda_1 = \lambda_1^0$, i.e. the no-slip condition is fulfilled. The straight line $\lambda_1 = \lambda_1^0$ becomes, in the physical plane,

$$y = (\nu/\gamma)^{1/2}\lambda_1^0\, x \tag{3.22}$$

This means that the case $m = -1$ corresponds to viscous flow in a wedge between two planes. Quantity γ can, in this case, be found from the angle between the two planes. The solution can also be obtained directly from the Hammel solution [9] by neglecting the terms which are small at high Reynold's numbers.

4. Numerical results. The problem (2.11) can be solved numerically for any values of the parameter m. Fig. 1 shows the behavior of $\varphi_2'(\lambda_1)$ for various values of m.

Considering the behavior of $\varphi_2'(\lambda_1)$ for $m = -0.25$ and $m = -0.375$ we find, that at large λ_1 the function can be written as

$$\varphi_2'(\lambda_1) = C_1(m)\,\lambda_1^m \tag{4.1}$$

Function $u_0(y)$ will then become

$$u_0(y) = C_1(m)\,|y|^m \tag{4.2}$$

When $0 > m > -0.5$, the resulting flow in the physical plane appears to be diffuse. Asymptotic behavior of the velocity of this flow depends on the parameter m. The coefficient $C_1(m)$ decreases together with m, e.g.

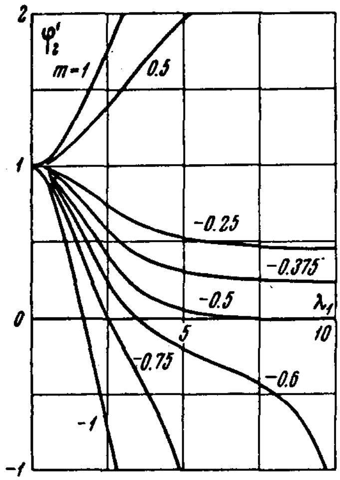

Fig. 1

$$C_1(-0.25) = 0.81; \quad C_1(-0.375) = 0.391$$

It appears that when $m = -0.5$ the coefficient C_1 becomes zero and the subsequent term of the asymptotic representation becomes the principal one (see (3.2)).

When $1 > m > 0$, we see from the example for $m = 0.5$ that the asymptotic behavior of $\varphi_2'(\lambda_1)$ can also be described by (4.1), when $\lambda_1 \to \infty$, while the function $u_0(y)$ is given by (4.2). In the physical plane this case corresponds to the mixing of two streams with parabolic profiles. Fig. 1 shows that at the values of m within the range $-0.5 > m > -1$ the function $\varphi_2'(\lambda_1) \to -\infty$ for $\lambda_1 \to \lambda_{1*}$ where λ_{1*} denotes some bounded limit value of λ_1. This makes the investigation of the asymptotic behavior of these functions as $\lambda_1 \to \infty$, impossible.

The cases with these values of m will correspond to flows in channels with curved walls

$$y = (\nu / \gamma)^{1/2} \lambda_1^0 x^{1/(m+2)} \tag{4.3}$$

Horizontal velocity at the walls will be equal to zero, while the vertical velocity will have some negative value corresponding to the influx through the walls.

5. Mixing in the presence of a pressure gradient. Let us consider a flow with mixing, in which the pressure is some function of x.

In this case the first equation of (1.1) will become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} \tag{5.1}$$

where $p = p(x)$ is the pressure referred to the density of the fluid. Self-similarity of the problem requires that the pressure is of the form

$$p = {}^1/_2\, \chi_0 x^{2m/(m+2)} \tag{5.2}$$

where χ_0 is a new pressure parameter. The equation of motion will now be

$$\nu\varphi''' + \frac{m+1}{m+2}\varphi\varphi'' - \frac{m}{m+2}(\varphi'^2 + \chi_0) = 0 \tag{5.3}$$

and on changing to the function $\varphi_2(\lambda_1)$ it will become

$$\varphi_2''' + \frac{m+1}{m+2}\varphi_2\varphi_2'' - \frac{m}{m+2}\left(\varphi_2'^2 + \frac{\chi_0}{\gamma^2}\right) = 0 \tag{5.4}$$

with the previous boundary conditions retained (see (2.8)).

Solution of this problem depends on the values of its two parameters, m and χ_0/γ^2.

Let us analyze the flows with the pressure gradient, for the specific values of m discussed in Section 3.

When $m = -0.5$, Eq. (5.4) can no longer be integrated to yield the exact solution, nor can its asymptotic behavior be obtained at $\lambda_1 \to \infty$. However a numerical solution is feasible. Fig. 2 shows the result of computing $\varphi_2'(\lambda_1)$ for various values of the ratio χ_0/γ^2 within the range $0 \geqslant \chi_0/\gamma^2 \geqslant -1$. We see from this figure that $\varphi_2'(\lambda_1) \to$ const when $\lambda_1 \to \infty$. When $\chi_0 / \gamma^2 = 0$, we obtain a well known exact solution. We also obtain the exact solution $\varphi_2(\lambda_1) = \lambda_1$ for all m, when $\chi_0 / \gamma^2 = -1$.

When $m = 1$, Eq. (5.4) is easily reduced to (3 7). The only change occurs in the value of $\varphi_2'''(0)$, which will now be $\varphi_2'''(0) = \varepsilon / 3$, where $\varepsilon = 1 + \chi_0 / \gamma^2$. Assuming that $\varepsilon > 0$ and proceeding as in Section 3, we can obtain the asymptotic behavior of the solution $\varphi_2(\lambda_1)$ in the presence of a pressure gradient. Integral equation analogous to (3. 8) will now be

$$g(\lambda_1) = \exp\left(-\frac{\lambda_1^2}{3}\right) \exp\left[-\frac{\varepsilon}{27}\int_0^{\lambda_1} g(\xi)(\lambda_1 - \xi)^3 d\xi\right] \qquad \left(g(\lambda_1) = \frac{3}{\varepsilon}\varphi_2'''(\lambda_1)\right) \qquad (5.5)$$

and the estimate (3. 10) will become

$$\Phi(h) - \Phi(q) \leqslant \frac{\varepsilon A}{3e^2} \qquad (5.6)$$

Relations (3. 14) – (3. 18) will remain the same, but the constants a and b will undergo a significant change and will now depend on χ_0 / γ^2.

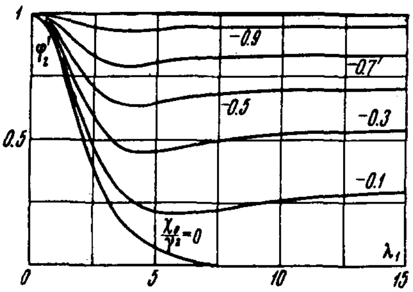

Fig. 2

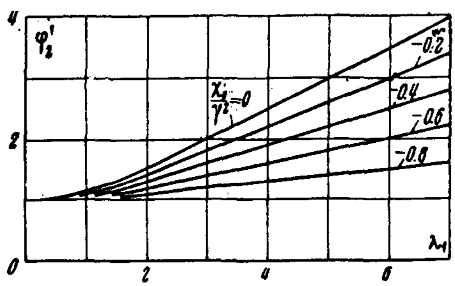

Fig. 3

Fig. 3 shows how the profile $\varphi_2'(\lambda_1)$ varies with varying χ_0 / γ^2.

This corresponds to the physical flow pattern already discussed for $p = \text{const}$. We see from Fig. 3 that for the values of χ_0 / γ^2 varying with the range $0 > \chi_0 / \gamma^2 > -1$, increase in the absolute value of the pressure leads to the straightening of the velocity profile. This follows from the fact that the pressure gradient is opposite to the velocity direction!

The case $m = 0$ is identical with that of Section 3.

When $m = -1$, we introduce the function $w = \varphi_2'$ to obtain the problem

$$w'' + w^2 + \chi_0 / \gamma^2 = 0, \qquad w(0) = 1, \qquad w'(0) = 0 \qquad (5.7)$$

whose solution reduces to an elliptic integral

$$\lambda_1 = \frac{\sqrt{6}}{2}\int_w^1 \frac{d\xi}{\sqrt{(1-\xi)(\xi-\xi_1)(\xi-\xi_2)}} \qquad \left(\xi_{1,2} = -\frac{1}{2} \pm \left[-3\left(\frac{1}{4} + \frac{\chi_0}{\gamma^2}\right)\right]^{1/2}\right) \qquad (5.8)$$

As before, we can find such λ_1^0, for which $w = 0$. However, λ_1^0 in this case will exist only for these values of χ_0/γ^2 for which the inequality

$$1/3\,(1 - \xi^3) + (1 - \xi)\ \chi_0 / \gamma^2 > 0 \quad (5.9)$$

holds for all ξ within the interval [0, 1]. Hence $\chi_0 / \gamma^2 > -1/3$. In the physical plane this case with $0 > \chi_0 / \gamma^2 > -1/3$ will correspond to the flow in a rectilinear divergent channel; λ_1^0 will now depend on the ratio χ_0 / γ^2 and both, the angle of inclination of the wall of the channel in the physical plane and the pressure $p(x)$, will have to be known in order to determine the horizontal velocity profile.

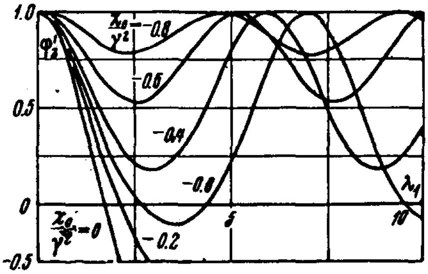

Fig. 4

Integral (5. 8) can easily be reduced to the standard form (see [10]).

If $0 > \chi_0 / \gamma^2 > -{}^1/_4$, then (5.10)

$$\lambda_1 = ({}^2/_3 t)^{-1/_2} F\left(2 \operatorname{arc\,ctg} \sqrt{(1-w)\, t^{-1}},\ \sqrt{{}^1/_2 + {}^3/_4\, t^{-1}}\right) \quad \left(t = \sqrt{3\,(1 + \chi_0 / \gamma^2)}\right)$$

If $-{}^1/_4 > \chi_0 / \gamma^2 > -1$, then

$$\lambda_1 = \left(\frac{6}{1-\xi_2}\right)^{1/_2} F(\mu, r), \quad F(\mu, r), \tag{5.11}$$

$$\mu = \arcsin\left(\frac{1-w}{1-\xi_1}\right)^{1/_2}, \qquad r = \left(\frac{1-\xi_1}{1-\xi_2}\right)^{1/_2}$$

The latter requires the restriction $1 > w \geqslant \xi_1 > \xi_2$, from which we find that

$$1 \geqslant \varphi_2'(\lambda_1) \geqslant -{}^1/_2 + \sqrt{-3\,({}^1/_4 + \chi_0 / \gamma^2)} \tag{5.12}$$

The behavior of $\varphi_2'(\lambda_1)$, in the two cases mentioned above is shown in Fig. 4. We find that when $-{}^1/_4 > \chi_0 / \gamma^2 > -1$, then $\varphi_2'(\lambda_1)$ is periodic. Minimum value of $\varphi_2'(\lambda_1)$ can be found from (5. 12).

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